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Construction of non-Abelian gauge theories on noncommutative spaces

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Abstract. We present a formalism to explicitly construct non-Abelian gauge theories on noncommutative spaces (induced via a star product with a constant Poisson tensor) from a consistency relation. This results in an expansion of the gauge parameter, the noncommutative gauge potential and fields in the fundamental representation, in powers of a parameter of the noncommutativity. This allows the explicit construction of actions for these gauge theories.

1 Introduction

Gauge theories can be formulated on noncommutative spaces. One recent approach is based on the Seiberg-Witten map [1]. This is the one we are particularly interested in because it allows the formulation of a Lagrangian theory in terms of ordinary fields. One can express such a theory very intuitively via covariant coordinates [2]. In this paper we give an explicit construction for the case of non-Abelian gauge groups. In contrast to earlier approaches [3], this method now works for arbitrary gauge groups.

The idea is to formulate field theories on noncommutative spaces as theories on commutative spaces and to express the noncommutativity by an appropriate $\star$-product. Gauge transformations then involve the $\star$-product as well. This prevents the gauge transformations from being Lie algebra-valued. They can, however, be defined in the enveloping algebra [4]. It is possible to find transformations representing the gauge group in the enveloping algebra that depend on the parameters and the gauge potential of the usual gauge theory only. An explicit form of such transformations can be constructed by a power series expansion in a parameter that characterizes the noncommutativity.

In the same manner fields that have the desired $\star$-product transformation properties can be constructed in terms of fields with the transformation properties of a usual gauge theory. For the gauge potential this amounts to the analogue of the Seiberg-Witten map for arbitrary non-Abelian gauge theories.

Finally we can consider actions that are invariant under the $\star$-product transformation laws. The Lagrangian of such an action can then be expanded in terms of the fields of a usual gauge theory and the parameter of the noncommutativity enters as a coupling constant. New coupling terms for a gauge theory appear. Such Lagrangians can be seen as effective Lagrangians that are meaningful in the tree approximation for the description of a phenomenological S-matrix. But one can also take them serious as Lagrangians for a quantum field theory with all the radiative corrections. In this context, the renormalizability of the theory has to be investigated [5].

In this paper we explicitly compute the formulas up to second order in the parameter that characterizes the noncommutativity in the case of the Moyal-Weyl product.

2 Gauge transformations

A non-Abelian gauge theory is based on a Lie algebra

$$[T^a, T^b] = i f^{ab}{}_c T^c. \tag{2.1}$$

In the usual formulation of a gauge theory fields are considered that transform under gauge transformations with Lie algebra-valued infinitesimal parameters[1]:

$$\delta_\alpha \psi^0(x) = i\alpha(x)\psi^0(x), \qquad \alpha(x) = \alpha_a(x) T^a. \tag{2.2}$$

It follows from (2.1) that

$$(\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha)\psi^0(x) = i\alpha_a(x)\beta_b(x) f^{ab}{}_c T^c \psi^0(x) \equiv \delta_{\alpha\times\beta}\psi^0(x), \tag{2.3}$$

with the shorthand

$$\alpha \times \beta \equiv \alpha_a \beta_b f^{ab}{}_c T^c = -i[\alpha, \beta]. \tag{2.4}$$

[1] Throughout we will denote fields with this transformation property by ψ^0

In addition, a gauge potential $a_{i,a}(x)$ is introduced with the transformation property

$$\delta_\alpha a_{i,a}(x) = \partial_i \alpha_a(x) - f^{bc}{}_a \alpha_b(x) a_{i,c}(x), \tag{2.5}$$

or equivalently,

$$\begin{aligned} a_i(x) &= a_{i,a}(x) T^a \\ \delta_\alpha a_i(x) &= \partial_i \alpha(x) + i[\alpha(x), a_i(x)]. \end{aligned} \tag{2.6}$$

This allows the definition of covariant derivatives and the field strength:

$$\begin{aligned} \mathcal{D}_i \psi^0(x) &= (\partial_i - i a_i)\psi^0(x) \\ F^0_{ij}\psi^0(x) &= i\big(\mathcal{D}_i\mathcal{D}_j - \mathcal{D}_j\mathcal{D}_i\big)\psi^0(x). \end{aligned} \tag{2.7}$$

In a gauge theory on noncommutative coordinates (2.2) is replaced by

$$\delta_\Lambda \psi(x) = i\Lambda(x) \star \psi(x). \tag{2.8}$$

The $\star$-product based on a quite general coordinate algebra has been defined in [4]. In this paper, we shall evaluate the respective formulas for the Moyal-Weyl-product only. This is the product that is most frequently discussed in the current literature, but we emphasize that the methods used in this paper work for other $\star$-products as well.

The $\star$-product of two functions does not commute, it reflects the algebraic properties of the space coordinates. As a consequence, two transformations of the type (2.8) cannot be reduced to the matrix commutator of the generators of the Lie algebra:

$$\begin{aligned} &(\delta_\Lambda \delta_\Sigma - \delta_\Sigma \delta_\Lambda)\psi(x) \\ &\quad = \big(\Lambda(x) \star \Sigma(x) - \Sigma(x) \star \Lambda(x)\big) \star \psi(x) \\ &\quad \equiv [\Lambda(x) \stackrel{\star}{,} \Sigma(x)] \star \psi(x). \end{aligned} \tag{2.9}$$

The parameters cannot be Lie algebra-valued, they have to be in the enveloping algebra:

$$\begin{aligned} \Lambda(x) = \Lambda_a(x) T^a &+ \Lambda^1_{ab}(x) : T^a T^b : + \dots \\ &+ \Lambda^{n-1}_{a_1\dots a_n}(x) : T^{a_1} \dots T^{a_n} : + \dots \quad . \end{aligned} \tag{2.10}$$

The dots indicate that we have to sum over a basis of the vector space spanned by the homogeneous polynomials in the generators of the Lie algebra. Completely symmetrized products form such a basis:

$$\begin{aligned} : T^a : &= T^a \\ : T^a T^b : &= \frac{1}{2}\{T^a, T^b\} = \frac{1}{2}(T^a T^b + T^b T^a) \\ : T^{a_1} \dots T^{a_n} : &= \frac{1}{n!} \sum_{\pi \in S_n} T^{a_{\pi(1)}} \dots T^{a_{\pi(n)}}. \end{aligned} \tag{2.11}$$

The $\star$-commutator of two enveloping algebra-valued transformations will remain enveloping algebra-valued. The price we seem to have to pay are infinitely many parameters $\Lambda^{n-1}_{a_1\dots a_n}(x)$, however, it is possible to define gauge transformations where all these infinitely many parameters depend on the usual gauge parameter $\alpha(x)$ and the gauge potential $a_i(x)$ and on their derivatives. Transformations of this type will be denoted $\Lambda_\alpha[a]$ and their x-dependence is purely via this finite set of parameters and gauge potentials $\Lambda_\alpha[a] \equiv \Lambda_{\alpha(x)}[a(x)]$ (for constant θ).

The transformation (2.8) will now be restricted to such parameters $\Lambda_\alpha[a]$

$$\delta_\alpha \psi(x) = i\Lambda_\alpha[a] \star \psi(x). \tag{2.12}$$

Each finite set of parameters $\alpha_a(x)$ defines a "tower" $\Lambda_\alpha[a]$ in the enveloping algebra that is entirely determined by the Lie algebra-valued part. To define and construct this tower we demand in accord with (2.3) (cf. [4]):

$$(\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha)\psi(x) = \delta_{\alpha\times\beta}\psi(x). \tag{2.13}$$

The variations δ_α are those of (2.12). More explicitly:

$$\begin{aligned} i\delta_\alpha \Lambda_\beta[a] - i\delta_\beta \Lambda_\alpha[a] + \Lambda_\alpha[a] \star \Lambda_\beta[a] - \Lambda_\beta[a] \star \Lambda_\alpha[a] \\ = i\Lambda_{\alpha\times\beta}[a]. \end{aligned} \tag{2.14}$$

The variation $\delta_\beta \Lambda_\alpha[a]$ refers to the a_i-dependence of $\Lambda_\alpha[a]$ and the transformation property (2.5) of a_i.

It is natural to expand the star product in its "noncommutativity" and to solve (2.14) in a power series expansion. For this purpose we introduce a parameter h:

$$\begin{aligned} \big(f \star g\big)(x) &= e^{\frac{i}{2}h \frac{\partial}{\partial x^i}\theta^{ij}\frac{\partial}{\partial y^j}} f(x) g(y)|_{y\to x} \\ &= f(x)g(x) + \frac{i}{2}h\theta^{ij}\partial_i f(x)\partial_j g(x) \\ &\quad - \frac{1}{8}h^2\theta^{ij}\theta^{kl}\partial_i\partial_k f(x)\partial_j\partial_l g(x) + \dots \quad . \end{aligned} \tag{2.15}$$

We could have used θ as an expansion parameter, however a θ-dependence of the fields might and will in fact arise for other reasons.

We assume that it is possible to expand the tower $\Lambda_\alpha[a]$ in the parameter h:

$$\Lambda_\alpha[a] = \alpha + h\Lambda^1_\alpha[a] + h^2\Lambda^2_\alpha[a] + \cdots \quad . \tag{2.16}$$

Now we expand (2.14) in h. To zeroth order we find (2.3) which has the solution (2.2). To first order we obtain

$$\begin{aligned} i\delta_\alpha \Lambda^1_\beta[a] - i\delta_\beta \Lambda^1_\alpha[a] + [\alpha, \Lambda^1_\beta[a]] - [\beta, \Lambda^1_\alpha[a]] - i\Lambda^1_{\alpha\times\beta}[a] \\ = -\frac{i}{2}\theta^{ij}\{\partial_i\alpha, \partial_j\beta\}. \end{aligned} \tag{2.17}$$

There is a homogeneous part in $\Lambda^1_\alpha[a]$ and an inhomogeneous part. We solve the inhomogeneous part with an ansatz linear in θ, because the inhomogeneous part is linear in θ as well. For dimensional reasons there is only a finite number of terms that can be used in such an ansatz. The proper combination of such terms is

$$\Lambda^1_\alpha[a] = \frac{1}{4}\theta^{ij}\{\partial_i\alpha, a_j\} = \frac{1}{2}\theta^{ij}\partial_i\alpha_a a_{j,b} : T^a T^b : . \tag{2.18}$$

The derivative term $\partial_i\alpha$ in the variation of the gauge potential (2.6) compensates the inhomogeneous part in (2.17), whereas the commutator term of the variation of the gauge potential combines with other terms of (2.17) to produce $\Lambda^1_{\alpha\times\beta}[a]$.

We can add solutions of the homogeneous part of (2.17). If there is a quantity F_i with the covariant transformation property

$$\delta_\alpha F_i = i[\alpha, F_i] \qquad (2.19)$$

and such a term can easily be constructed, for instance from the field strength and the covariant derivative in (2.7)

$$F_i = \theta^{jk}\mathcal{D}_j F^0_{ki}, \qquad (2.20)$$

then we find a solution $\widetilde{\Lambda}^1_\alpha[a]$ of the homogeneous part of (2.17):

$$\widetilde{\Lambda}^1_\alpha[a] = c\theta^{ij}\{\partial_i\alpha, F_j\}. \qquad (2.21)$$

This term can be added to $\Lambda^1_\alpha[a]$ defined in (2.18). To first order in h we obtain

$$\Lambda^1_\alpha[a] = \theta^{ij}\left\{\partial_i\alpha, \frac{1}{4}a_j + cF_j\right\}. \qquad (2.22)$$

We can view the additional term as a redefinition of the gauge potential:

$$\widetilde{a_i} = a_i + 4cF^\alpha_i. \qquad (2.23)$$

It does not change the transformation properties (2.6)

$$\delta_\alpha\widetilde{a_i} = \partial_i\alpha + i[\alpha, \widetilde{a_i}]. \qquad (2.24)$$

Such generalized solutions as (2.22) have to be expected, as it is only the transformation property of a_i that matters. To define a physical theory, we have to make use of the full freedom of the Seiberg-Witten map. It has been shown that for finding a renormalizable theory the extra terms are essential [6].

There are other solutions of the homogeneous equation that cannot be obtained from a redefinition of the vector potential a_i. An example is

$$\widetilde{\Lambda}^{1\,\prime}_\alpha[a] = c\theta^{ij}[\partial_i\alpha, a_j]. \qquad (2.25)$$

The choice of the constant $c = -\frac{1}{4}$ for this solution of the homogeneous part of (2.17) would simplify some of the calculations to come, however, we decide to work with (2.18) instead, since this is a solution expressed in the completely symmetrized basis (2.11) in the generators of the Lie algebra.

To second order in h we obtain from (2.14):

$$\begin{aligned} &i\delta_\alpha\Lambda^2_\beta[a] - i\delta_\beta\Lambda^2_\alpha[a] + [\alpha, \Lambda^2_\beta[a]] - [\beta, \Lambda^2_\alpha[a]] - i\Lambda^2_{\alpha\times\beta}[a] \\ &\quad = +\frac{1}{8}\theta^{ij}\theta^{kl}[\partial_i\partial_k\alpha, \partial_j\partial_l\beta] \\ &\quad -\frac{i}{2}\theta^{ij}\Big(\{\partial_i\Lambda^1_\alpha[a], \partial_j\beta\} - \{\partial_i\Lambda^1_\beta[a], \partial_j\alpha\}\Big) \\ &\quad -[\Lambda^1_\alpha[a], \Lambda^1_\beta[a]]. \end{aligned} \qquad (2.26)$$

The homogeneous part of the equation has the same structure as before. We shall use the expression (2.18) for $\Lambda^1_\alpha[a]$ and we see that the terms of the inhomogeneous part involving $\Lambda^1_\alpha[a]$ contribute to third order in T^a. With an appropriate ansatz we can eliminate all these terms of third order and of second order in T^a as well. The respective terms in the solution (2.27) can easily be identified. Finally we obtain a solution of (2.26)[2]:

$$\begin{aligned} \Lambda^2_\alpha[a] = \frac{1}{32}\theta^{ij}\theta^{kl}\Big(&-4\{\partial_i\alpha, \{a_k, \partial_l a_j\}\} \\ &-i\{\partial_i\alpha, \{a_k, [a_j, a_l]\}\} - i\{a_j, \{a_l, [\partial_i\alpha, a_k]\}\} \\ &+2i[\partial_i\partial_k\alpha, \partial_j a_l] - 2[\partial_j a_l, [\partial_i\alpha, a_k]] \\ &+2i[[a_j, a_l], [\partial_i\alpha, a_k]]\Big). \end{aligned} \qquad (2.27)$$

The solutions (2.18) and (2.27) are such that they are of first and second order in θ respectively. We know from [4] that we can expect a solution of (2.14) where the order in θ and the order in T^a are related. In such a solution the contribution in θ^n will be of order $n+1$ in T^a. The above solutions are of this type. This can however be changed by adding θ-dependent solutions of the homogeneous equation.

3 Fields

In a usual gauge theory, fields have the transformation property (2.2). We have denoted fields that transform this way by ψ^0. In a gauge theory with the $\star$-product fields are supposed to transform as in (2.12). We show that fields with this transformation property can be built from fields with the transformation property (2.2) and the gauge potential a_i.

We expand in powers of h:

$$\psi[a] = \psi^0 + h\psi^1[a] + h^2\psi^2[a] + \cdots \quad . \qquad (3.1)$$

To zeroth order in h, we obtain (2.2) and to first order:

$$\delta_\alpha\psi^1[a] = i\alpha\psi^1[a] + i\Lambda^1_\alpha[a]\psi^0 - \frac{1}{2}\theta^{ij}\partial_i\alpha\partial_j\psi^0. \qquad (3.2)$$

If we take the solution (2.18) for $\Lambda^1_\alpha[a]$, we find that

$$\psi^1[a] = -\frac{1}{2}\theta^{ij}a_i\partial_j\psi^0 + \frac{i}{4}\theta^{ij}a_i a_j\psi^0. \qquad (3.3)$$

will have the desired transformation property (2.12) to first order in h. We proceed to the next order,

$$\begin{aligned} \delta_\alpha\psi^2[a] = &\; i\alpha\psi^2[a] + i\Lambda^1_\alpha[a]\psi^1[a] + i\Lambda^2_\alpha[a]\psi^0 \\ &-\frac{1}{2}\theta^{ij}\partial_i\Lambda^1_\alpha[a]\partial_j\psi^0 - \frac{1}{2}\theta^{ij}\partial_i\alpha\partial_j\psi^1[a] \\ &-\frac{i}{8}\theta^{ij}\theta^{kl}\partial_i\partial_k\alpha\partial_j\partial_l\psi^0, \end{aligned} \qquad (3.4)$$

use (2.27) for $\Lambda^2_\alpha[a]$ and find:

$$\psi^2[a] = \frac{1}{32}\theta^{ij}\theta^{kl}\Big(-4i\partial_i a_k\partial_j\partial_l\psi^0 + 4a_i a_k\partial_j\partial_l\psi^0$$

[2] Similar results have been obtained in [7] and [8] in the context of $U(n)$

$$+8a_i\partial_j a_k\partial_l\psi^0 - 4a_i\partial_k a_j\partial_l\psi^0 - 4ia_ia_ja_k\partial_l\psi^0$$
$$+4ia_ka_ja_i\partial_l\psi^0 - 4ia_ja_ka_i\partial_l\psi^0 + 4\partial_j a_ka_i\partial_l\psi^0$$
$$-2\partial_i a_k\partial_j a_l\psi^0 + 4ia_ia_l\partial_k a_j\psi^0 + 4ia_i\partial_k a_ja_l\psi^0$$
$$-4ia_i\partial_j a_ka_l\psi^0 + 3a_ia_ja_la_k\psi^0 + 4a_ia_ka_ja_l\psi^0$$
$$+2a_ia_la_ka_j\psi^0\Big). \quad (3.5)$$

Homogeneous solutions of (3.2) and (3.2) can naturally be added to these solutions. The adjoint field $\bar{\psi}[a]$ is easily obtained from (3.3) and (3.5), keeping in mind that a_i is supposed to be self-adjoint for a unitary gauge group.

4 Gauge potentials and field strengths

In the same way as in the last section for ordinary fields we can solve for an enveloping algebra-valued gauge potential. Its transformation property is (for a definition, e.g. [2]):

$$\delta_\alpha A_i = \partial_i\Lambda_\alpha[a] + i[\Lambda_\alpha[a] \stackrel{\star}{,} A_i]. \quad (4.1)$$

Again we expand in h:

$$A_i[a] = a_i + hA_i^1[a] + h^2A_i^2[a] + \cdots \quad . \quad (4.2)$$

As expected, we can recapture (2.6) to zeroth order, to first order we obtain:

$$\delta_\alpha A_i^1[a] = \partial_i\Lambda_\alpha^1[a] + i[\Lambda_\alpha^1[a], a_i] + i[\alpha, A_i^1[a]]$$
$$-\frac{1}{2}\theta^{kl}\{\partial_k\alpha, \partial_l a_i\}. \quad (4.3)$$

Again we use the solution (2.18) for $\Lambda_\alpha^1[a]$ and find as a solution to (4.3)

$$A_i^1[a] = -\frac{1}{4}\theta^{kl}\{a_k, \partial_l a_i + F_{li}^0\}, \quad (4.4)$$

where F_{ij}^0 is the field strength of the ordinary Lie algebra-valued gauge theory, introduced in (2.7)

$$F_{ij}^0 = \partial_i a_j - \partial_j a_i - i[a_i, a_j]. \quad (4.5)$$

To second order in h we obtain from (4.1):

$$\delta_\alpha A_i^2[a] = \partial_i\Lambda_\alpha^2[a] + i[\alpha, A_i^2[a]] + i[\Lambda_\alpha^1[a], A_i^1[a]]$$
$$+i[\Lambda_\alpha^2[a], a_i] - -\frac{1}{2}\theta^{kl}\{\partial_k\alpha, \partial_l A_i^1[a]\}$$
$$-\frac{1}{2}\theta^{kl}\{\partial_k\Lambda_\alpha^1[a], \partial_l a_i\}$$
$$-\frac{i}{8}\theta^{kl}\theta^{mn}[\partial_k\partial_m\alpha, \partial_l\partial_n a_i]. \quad (4.6)$$

With the choice (2.27) for $\Lambda_\alpha^2[a]$ this has the following solution:

$$A_i^2[a] = \frac{1}{32}\theta^{kl}\theta^{mn}\Big(4i[\partial_k\partial_m a_i, \partial_l a_n] - 2i[\partial_k\partial_i a_m, \partial_l a_n]$$
$$+4\{a_k, \{a_m, \partial_n F_{li}^0\}\} + 2[[\partial_k a_m, a_i], \partial_l a_n]$$
$$-4\{\partial_l a_i, \{\partial_m a_k, a_n\}\} + 4\{a_k, \{F_{lm}^0, F_{ni}^0\}\}$$
$$-i\{\partial_i a_n, \{a_l, [a_m, a_k]\}\} - i\{a_m, \{a_k, [\partial_i a_n, a_l]\}\}$$
$$+4i[[a_m, a_l], [a_k, \partial_n a_i]] - 2i[[a_m, a_l], [a_k, \partial_i a_n]]$$
$$-\{a_m, \{a_k, [a_l, [a_n, a_i]]\}\}$$
$$+\{a_k, \{[a_l, a_m], [a_n, a_i]\}\}$$
$$+[[a_m, a_l], [a_k, [a_n, a_i]]]\Big). \quad (4.7)$$

With this solution at hand we now turn to the enveloping algebra-valued field strength (defined in [2]):

$$F_{ij} = \partial_i A_j - \partial_j A_i - i[A_i \stackrel{\star}{,} A_j], \quad (4.8)$$

with the transformation property

$$\delta_\alpha F_{ij} = i[\Lambda_\alpha[a] \stackrel{\star}{,} F_{ij}]. \quad (4.9)$$

To express this field strength, we insert (4.4) and (4.7) into (4.8). We could have used (4.9) to find a field with the desired transformation property, as we did in Sect. 3. This however does not reproduce the full solution (4.10) which rests on the definition of $F_{ij}[a]$ in terms of the gauge potential (4.2):

$$F_{ij}^1[a] = \frac{1}{2}\theta^{kl}\{F_{ik}^0, F_{jl}^0\} - \frac{1}{4}\theta^{kl}\{a_k, (\partial_l + \mathcal{D}_l)F_{ij}^0\}. \quad (4.10)$$

$F_{ij}^2[a]$ to second order in h is obtained similarly by inserting (4.7) into (4.8). The gauge potential A_i with its transformation property (4.1) allows the definition of a covariant derivative

$$\mathcal{D}_i\psi = \partial_i\psi - iA_i \star \psi \quad (4.11)$$

with the transformation (2.12):

$$\delta_\alpha\mathcal{D}_i\psi = i\Lambda_\alpha[a] \star \mathcal{D}_i\psi. \quad (4.12)$$

5 Actions

The transformation laws of the field strength (4.9), the fields (2.12) and the covariant derivatives (4.12) allow the construction of invariant actions. It can be shown by partial integration that the integral has the trace property for the $\star$-product:

$$\int f \star g \;\; \mathrm{d}x = \int g \star f \;\; \mathrm{d}x = \int fg \;\; \mathrm{d}x. \quad (5.1)$$

Thus we find an invariant action for the gauge potential

$$S = -\frac{1}{4}\mathrm{Tr}\int F_{ij} \star F^{ij} \;\; \mathrm{d}x, \quad (5.2)$$

as well as for the matter fields

$$S = \int \bar{\psi} \star (\gamma^i\mathcal{D}_i - m)\psi \;\; \mathrm{d}x. \quad (5.3)$$

Our aim is to expand these actions in the fields a_i and ψ^0 and to treat them as conventional field theories depending on a coupling constant θ. We only do this here to first order in h and construct the Lagrangian from our previous results:

$$\begin{aligned} m\bar{\psi}\star\psi &= m\bar{\psi}^0\psi^0+\frac{i}{2}h\theta^{kl}m\mathcal{D}_k\bar{\psi}^0\mathcal{D}_l\psi^0\\ \bar{\psi}\star\gamma^i\mathcal{D}_i\psi &= \bar{\psi}^0\gamma^i\mathcal{D}_i\psi^0+\frac{i}{2}h\theta^{kl}\mathcal{D}_k\bar{\psi}^0\gamma^i\mathcal{D}_l\mathcal{D}_i\psi^0\\ &\quad -\frac{1}{2}h\theta^{kl}\bar{\psi}^0\gamma^iF^0_{ik}\mathcal{D}_l\psi^0\\ F_{ij}\star F^{ij} &= F^0_{ij}F^{0ij}+\frac{i}{2}h\theta^{kl}\mathcal{D}_kF^0_{ij}\mathcal{D}_lF^{0ij}\\ &\quad +\frac{1}{2}h\theta^{kl}\{\{F^0_{ik},F^0_{jl}\},F^{0ij}\}\\ &\quad -\frac{1}{4}h\theta^{kl}\{F^0_{kl},F^0_{ij}F^{0ij}\}\\ &\quad -\frac{i}{4}h\theta^{kl}[a_k,\{a_l,F^0_{ij}F^{0ij}\}] \end{aligned} \tag{5.4}$$

For the action we use partial integration and the cyclicity of the trace and obtain to first order in h:

$$\begin{aligned} &\int\bar{\psi}\star(\gamma^i\mathcal{D}_i-m)\psi\,\mathrm{d}x\\ &=\int\bar{\psi}^0(\gamma^i\mathcal{D}_i-m)\psi^0\,\mathrm{d}x\\ &\quad-\frac{1}{4}h\theta^{kl}\int\bar{\psi}^0F^0_{kl}(\gamma^i\mathcal{D}_i-m)\psi^0\,\mathrm{d}x\\ &\quad-\frac{1}{2}h\theta^{kl}\int\bar{\psi}^0\gamma^iF^0_{ik}\mathcal{D}_l\psi^0\,\mathrm{d}x \end{aligned} \tag{5.5}$$

$$\begin{aligned} &-\frac{1}{4}\mathrm{Tr}\int F_{ij}\star F^{ij}\,\mathrm{d}x\\ &=-\frac{1}{4}\mathrm{Tr}\int F^0_{ij}F^{0ij}\,\mathrm{d}x+\frac{1}{16}h\theta^{kl}\mathrm{Tr}\int F^0_{kl}F^0_{ij}F^{0ij}\,\mathrm{d}x\\ &\quad-\frac{1}{2}h\theta^{kl}\mathrm{Tr}\int F^0_{ik}F^0_{jl}F^{0ij}\,\mathrm{d}x \end{aligned} \tag{5.6}$$

6 The Abelian case

Noncommutative Abelian gauge theories have recently been studied intensively and substantial results have been obtained.

If such a theory is expanded not in the noncommutativity h as in the previous chapters, but in powers of the gauge potential of the commutative theory as suggested in [9], the following result is valid to all orders in θ and first non-trivial order in a:

$$A^i[a]=\theta^{ij}\big(a_j+\frac{1}{2}\theta^{kl}a_l\star_2(\partial_ka_j+f_{kj})+\cdots\big) \tag{6.1}$$

$$\Lambda_\alpha[a]=\alpha+\frac{1}{2}\theta^{kl}a_l\star_2\partial_k\alpha+\cdots\quad, \tag{6.2}$$

where $f_{jk}=\partial_ja_k-\partial_ka_j$ is the Abelian field strength and $\star_2$ is an abbreviation for the following power series in the noncommutativity[3] (it is not a $\star$-product though):

$$f\star_2 g=\mu\left(\frac{\sin(\frac{h\theta^{ij}}{2}\partial_i\otimes\partial_j)}{\frac{h\theta^{ij}}{2}\partial_i\otimes\partial_j}\right)(f\otimes g), \tag{6.3}$$

and $\mu(f\otimes g)=f\cdot g$ the ordinary multiplication map. It is particularly convenient to use this multiplication in the Fourier representation.

We will now derive this result. We know from [4] and [12] that the following expressions for the noncommutative gauge potential and gauge parameter satisfy both the Seiberg-Witten gauge condition and the consistency relation (these expressions are valid for arbitrary Poisson structures $\theta(x)$):

$$A^i[a]=\big(\exp(a_\star+\partial_t)-1\big)\,x^i \tag{6.4}$$

$$\Lambda_\alpha[a]=\Big(\frac{\exp(a_\star+\partial_t)-1}{a_\star+\partial_t}\Big)\,\alpha, \tag{6.5}$$

with the differential operator

$$a_\star=\sum\frac{(ih)^n}{n!}U_{n+1}(a_\theta,\theta,\cdots,\theta)=\theta^{ij}a_j\partial_i+\cdots\quad, \tag{6.6}$$

and the rule $\partial_t\theta^{il}=-\theta^{ij}f_{jk}\theta^{kl}$. The differential operator $a_\star$ is obtained from the vector field $a_\theta=\theta^{ij}a_j\partial_i$ and the Poisson bivector $\theta=\theta^{ij}\partial_i\otimes\partial_j$ via Kontsevich's formality maps U_n (for further clarifications we refer to the mentioned articles).

Expanding the exponentials results in an expansion in powers of the ordinary gauge potential a_i, each term containing *all* powers of h:

$$A^i[a]=\underbrace{a_\star x^i}_{\mathcal{O}(a^1)}+\underbrace{\frac{1}{2}(a_\star^2+\dot{a}_\star)x^i}_{\mathcal{O}(a^2)}+\cdots \tag{6.7}$$

$$\Lambda_\alpha[a]=\underbrace{\alpha}_{\mathcal{O}(a^0)}+\underbrace{\frac{1}{2}a_\star\alpha}_{\mathcal{O}(a^1)}+\cdots\quad. \tag{6.8}$$

Kontsevich has given a graphical prescription similar to Feynman diagrams to compute the formality maps. Using these it is straightforward to compute $a_\star$ explicitly to all orders in h for constant θ. The result is:

$$a_\star g=\big(\theta^{ij}a_j\big)\star_2\partial_ig=\theta^{ij}a_j\partial_ig+\cdots\quad, \tag{6.9}$$

with the already mentioned product $\star_2$. Inserting this $a_\star$ into (6.7) and using the fact that $a_\star x^i=a_\theta x^i$ gives the expressions for $A^i[a]$ and $\Lambda_\alpha[a]$ stated at the beginning of this section. Higher order terms can be obtained similarly. A nice heuristic derivation of these results based on the consistency condition has been given in [11].

7 Expansion of non-Abelian fields in a

Adopting a similar approach like in the previous section, we here state a straight-forward result concerning the expansion of fields in a non-Abelian gauge theory in powers of the commutative gauge potential.

[3] This notation $\star_2$ is now widely used, e.g. in [9], [10] and [11]

Assume that a field ψ (a representation of the enveloping algebra $\delta_\alpha \psi = i\Lambda_\alpha[a] \star \psi$) can be written as a matrix-valued differential operator $\Phi[a]$ applied to the field ψ^0 in the representation of the Lie algebra ($\delta_\alpha \psi^0 = i\alpha\psi^0$): $\psi = \Phi[a]\psi^0$. Then the variation of ψ can be written in the following way:

$$\big(\delta_\alpha \Phi[a]\big)\psi^0 + \Phi[a]\big(i\alpha\psi^0\big) \stackrel{!}{=} i\Lambda_\alpha[a] \star \big(\Phi[a]\psi^0\big). \quad (7.1)$$

To zeroth order in a this reads:

$$\Phi^1[\partial\alpha]\psi^0 + i\alpha\psi^0 = i\alpha \star \psi^0. \quad (7.2)$$

The second term in the variation of a, $i[\alpha, a]$, drops out being first order in a. Due to the Bianchi identity of a non-Abelian gauge theory ($df + a \wedge f = 0$), this expansion is not well defined to higher orders in a and we will not discuss orders different from $\mathcal{O}(a^0)$. This problem does not occur for the Abelian case. Continuing in our analysis we set

$$\Phi^1[\partial\alpha]\psi^0 = i\alpha \star \psi^0 - i\alpha\psi^0 =: -\frac{h}{2}\theta^{kl}(\partial_k\alpha) \bullet \partial_l\psi^0, \quad (7.3)$$

where we have introduced the following shorthand[4]

$$f \bullet g = \mu \left(\frac{e^{\frac{ih}{2}\theta^{kl}\partial_k \otimes \partial_l} - 1}{\frac{ih}{2}\theta^{kl}\partial_k \otimes \partial_l} \right) (f \otimes g). \quad (7.4)$$

This should be compared with the Moyal-Weyl-product: $f \star g := \mu(e^{\frac{ih}{2}\theta^{kl}\partial_k \otimes \partial_l})(f \otimes g)$. With this shorthand for the product we are free to integrate:

$$\Phi^1[a_k]\psi^0 = -\frac{h}{2}\theta^{kl}(a_k) \bullet \partial_l\psi^0. \quad (7.5)$$

Therefore we obtain the following expansion (to first power in a and all powers in h):

$$\psi = \psi^0 - \frac{h}{2}\theta^{kl}(a_k) \bullet \partial_l\psi^0 + \cdots \quad , \quad (7.6)$$

Okuyama [11] has computed $A^i[a]$ and $\Lambda_\alpha[a]$ in a similar fashion.

[4] This product was also introduced in [13], there called $\star''$

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